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FORCED OSCILLATIONS OF NONLINEAR HAMILTONIAN SYSTEMS, II.(U)
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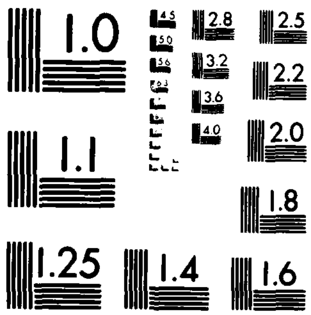

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(6) FORCED OSCILLATIONS OF NONLINEAR HAMILTONIAN SYSTEMS, II.

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FORCED OSCILLATIONS OF NONLINEAR HAMILTONIAN SYSTEMS, II

Ivar Ekeland*

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ABSTRACT

We study periodic solutions of the nonlinear Hamiltonian system with n degrees of freedom:

$$(\mathcal{H}) \qquad \begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i}(x,p) + g_i(t) \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i}(x,p) + h_i(t) \end{aligned}$$

the Hamiltonian H being convex and super quadratic in both variables, and the forcing terms being T-periodic with mean value zero. We prove that, if these forcing terms lie within bounds which we explicitly compute, system $(\mathcal{H})$ has some T-periodic solution, which we also locate explicitly.

AMS (MOS) Subject Classifications - 34C15, 34C25, 49A10, 70H30, 70K99

Key Words - Hamiltonian system, nonlinear oscillation

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SIGNIFICANCE AND EXPLANATION

This is the second in a series of two papers, the purpose of which is to study forced oscillations of a large class of nonlinear conservative systems. The novelty of the approach is that we do not suppose the oscillations to be small.

The systems we are interested in are described by equations of the following type:

$$(H) \qquad \ddot{x}_i + \frac{\partial V}{\partial x_i}(x_1,\ldots,x_n) = f_i(t) \quad 1 \le i \le n .$$

where the potential V is assumed to be convex with respect to all variables, and has the origin as an equilibrium point. A typical one-dimensional example is a spring: it will be linear if it follows Hooke's law, $F = a\ell$, sublinear if it follows the law $F = a\ell^\theta$ for some $\theta \in (0,1)$, and superlinear if its law is $F = a\ell^\theta$ for some $\theta > 1$.

In a previous report, in collaboration with F. Clarke, we studied equations (H) for sublinear systems, and found that T-periodic oscillations will exist for any T-periodic forcing terms f_i. This is in sharp contrast to the linear case, where the resonance values of T have to be excluded.

In this paper, the superlinear case is studied. It is found that T-periodic oscillations will exist provided the forcing terms f_i are T-periodic, have mean value zero, and fall within some range. This range (not necessarily small) is computed explicitly, and bounds are given for the oscillations.

FORCED OSCILLATIONS OF NONLINEAR HAMILTONIAN SYSTEMS, II

Ivar Ekeland*

§I - INTRODUCTION.

Hamilton's equations, for a system with n degrees of freedom, are:

$$(H) \qquad (-\dot{p},\dot{x}) - H'(t,x,p) = f(t) \in \mathbb{R}^{2n}$$

We refer to periodic solutions of (H) as oscillations. They are free if f is identically zero, forced otherwise. Of course, the forcing term f itself will be required to be periodic, although this by itself will not imply that (H) has a periodic solution.

This paper studies forced oscillations for a particular type of nonlinear Hamiltonian aystems. The origin will be an equilibrium, and the Hamiltonian itself will be convex and superquadratic in all variables (x,p) together. For instance, $H(x,p) = (\Sigma x_i^2 + \Sigma p_i^2)^\theta$, with $\theta > 1$, will do. An example of such a system is a taut spring,which does not follow Hooke's law, $F = k\ell$, force proportional to length, but the law $F = k\ell^\theta$ with $\theta > 1$.

Free oscillations for such systems were first studied by Rabinowitz ([8], [9]). The author obtained similar results ([6]), by using a variational method devised by F. Clarke and himself for convex subquadratic Hamiltonians ([2], [3]).

This paper, although self-contained, borrows heavily from the latter approach. It relies on a dual version of the least action principle, stated here as proposition 2.2, but which can be found also in the papers [3], [6], and particularly [4]. The associated variational problem is shown to have a local minimum (proposition 3.1), which gives rise to a periodic solution of the original Hamiltonian system.

Indeed, one can view this paper as a sequel to [ ], which treated forced oscillations for convex subquadratic Hamiltonians. It is interesting to note that, in the

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latter case, periodic solutions to (H) exist for any forcing term. This is no longer true for quadratic Hamiltonians, such as $\frac{1}{2}\Sigma p_i^2 + \frac{1}{2}\Sigma \alpha_i x_i^2$, because of the onset of resonance.

Nor is it true in the case we are dealing with, a superquadratic Hamiltonian. We are able to show the existence of a periodic solution to system ( ) only if the forcing term f has zero mean, and is smaller than some bound, not necessarily small, which we compute explicitly (propositions 3.1 and 4.2).

We do not know what happens beyond this bound. A more detailed analysis shows the following. The periodic solutions we find give local minima of the dual action integral, and converge towards the equilibrium solution when the forcing term goes to zero. On the other hand, with a few more assumptions on H, there will be another kind of periodic solutions, which correspond to saddle-points of the dual action integral, and which converge to a non-constant solution when the forcing term goes to zero. Now, when the forcing term increases, it may well be that these two kinds of periodic solution, which are well apart when f is small, begin interfering, and finally destroy each other. That, at least, is what the behaviour of the dual action integral would suggest.

§II - THE DUAL ACTION INTEGRAL.

We are investigating the differential system:

$$(H) \qquad \dot{u}(t) \in \sigma\partial H(t,u(t)) + f(t)$$

The function $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ is the Hamiltonian. In the two following sections, II and III, it will always be assumed that $H(t,u)$ is measurable in t, convex and continuous in u, with:

$$(1) \qquad H(t,u) \geq H(t,0) = 0 \quad \text{for all} \quad (t,u)$$

and that there are constants $k > 0$ and $\beta > 2$ such that:

$$(2) \qquad H(t,u) \leq \frac{k^\beta}{\beta} |u|^\beta \quad \text{for all} \quad (t,u)$$

$$(3) \qquad \forall t,\ r^{-1}\mathrm{Min}\{H(t,u) \mid |u| = r\} \to +\infty \quad \text{as} \quad r \to \infty .$$

The symbol $\partial H(t,u)$ denotes the subdifferential of the function $u' \to H(t,u')$ at the point $u \in \mathbb{R}^{2n}$, in the sense of convex analysis (see [10] or [7]). It is defined by:

$$(4) \qquad v \in \partial H(t,u) \Leftrightarrow \forall u',\ H(t,u') \geq H(t,u) + (u' - u,v)$$

We denote by σ a linear operator which, in some appropriate base of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ can be written as:

$$(5) \qquad \sigma = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$$

In other words, in that particular base, $u \in \mathbb{R}^{2n}$ is written $u = (x,p)$, and $\sigma(u) = (p,-x)$. The x_i, $1 \leq i \leq n$, are position variables, and the p_i, $1 \leq i \leq n$, momentum variables. We have of course ${}^t\sigma = \sigma^{-1} = -\sigma$.

The function $f : \mathbb{R} \to \mathbb{R}^{2n}$ is the forcing term. It will be assumed to be integrable over every bounded interval.

A solution of the differential inclusion (H) is an absolutely continuous function $u : \mathbb{R} \to \mathbb{R}^{2n}$, with derivative $\dot{u}$, such that relation (H) holds for almost every t. Of course, if f is continuous and H is differentiable in u, the derivative H' being continuous in (t,u), then u becomes a classical C^1 solution of the differential equation:

$$\dot{u} = \sigma H'(t,u) + f(t) \ .$$

Setting $u = (x,p)$ and $f = (g,h)$, we get the familiar form of the Hamiltonian system (H):

$$\begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial p_i}(t,x,p) + g_i(t) \\[2mm] \dot{p}_i = -\dfrac{\partial H}{\partial x_i}(t,x,p) + h_i(t) \ . \end{cases}$$

Our investigation of system (H) will rely on the dual approach initiated in [2] and developed in [3], [4], [6]. First, we have to introduce the Fenchel conjugate $G(t,\cdot)$ of the convex function $H(t,\cdot)$ (for the time being, t is just a parameter, pegged at some given value):

(7) $$G(t,v) = \operatorname{Sup}\{(v,u) - H(t,u) \mid u \in \mathbb{R}^{2n}\}$$

In the case where $H(t,\cdot)$ is differentiable, this definition reduces to the familiar formula for the Legendre transform: $G(t,v) = (v,u) - H(t,u)$, with $v = H'(t,u)$. Formula (7), however, will hold even in non-differentiable cases, with the following results:

<u>Lemma 2.1</u>. The function $G : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ is measurable in t, convex and continuous in v. It satisfies the following inequalities, where $\alpha = \dfrac{\beta}{\beta - 1}$ is the conjugate exponent of β:

(8) $$G(t,v) \geq G(t,0) = 0 \quad \text{for all} \quad (t,v)$$

(9) $$G(t,v) \geq \frac{1}{\alpha k^{\alpha}} |v|^{\alpha} \quad \text{for all} \quad (t,v)$$

Moreover, the three following relations are equivalent:

(10) $$u \in \partial G(t,v)$$

(11) $$v \in \partial H(t,u)$$

(12) $$G(t,v) + H(t,u) = (u,v)$$

<u>Proof</u>. Convexity and lower semi-continuity of $G(t,\cdot)$, as well as the Fenchel reciprocity formulas (10) $\Leftrightarrow$ (11) $\Leftrightarrow$ (12), are classical properties, which can be found in [10] or [7]. Measurability with respect to t is proved in [7], chapter 7, or in the paper [11]. We check the remaining properties by using the definition (7) of G.

Because of condition (2), the function $G(t,\cdot)$ is finite at every point of $\mathbb{R}^{2n}$. Since it also is convex, it must be continuous.

We have by condition (1):

$$G(t,v) \geq (v,0) - H(t,0) = 0$$

$$G(t,0) = \sup_u - H(t,u) = 0$$

Hence formula (8). Now for (9), using inequality (2)

$$\begin{aligned} G(t,v) &\geq \operatorname{Sup}\{(v,u) - \frac{k^\beta}{\beta}|u|^\beta | u \in \mathbb{R}^{2n}\} \\ &= \sup_{s\geq 0} \sup_{|u|=s} \{(v,u) - \frac{k^\beta}{\beta} s^\beta\} \\ &= \sup_{s\geq 0} \{s|v| - \frac{k^\beta}{\beta} s^\beta\} \\ &= \frac{1}{k^{\beta/(\beta-1)}} \left(1 - \frac{1}{\beta}\right)|v|^{\beta/(\beta-1)} \end{aligned}$$

The dual action integral can now be written:

$$I(v) = \int_0^T \left\{\frac{1}{2}(\sigma\dot{v}(t),v(t)) + G(t,-\sigma\dot{v}(t) + \sigma f(t))\right\}dt \tag{13}$$

It was introduced in [ ], where its critical points where related to solutions of system (H). We shall investigate it anew, both to make the paper self-contained and to account for some differences in the analytical setting.

We shall be working in the function spaces $L^\alpha(0,T;\mathbb{R}^{2n})$ and $L^\beta(0,T;\mathbb{R}^{2n})$, and denoting by $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ the corresponding norms. We are particularly interested in the Sobolev space $W^{1,\alpha}(0,T;\mathbb{R}^{2n})$, and its closed subspace E defined by:

$$v \in E \Leftrightarrow \dot{v} \in L^\alpha(0,T:\mathbb{R}^{2n}) \text{ and } \int_0^T \dot{v}(t)dt = 0 = \int_0^T v(t)dt \tag{14}$$

The norm of v in E will be defined to be $\|\dot{v}\|_\alpha$. The dual action integral (13) now defines a functional I on E. Our next results relates local minima of I to solutions of (H) satisfying $u(0) = u(T)$.

Note that the integrand in formula (13), i.e. the function

$$L(t,v,w) = \frac{1}{2}(\sigma w,v) + G(t,-\sigma w + \sigma f(t)) \tag{15}$$

on $\mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is neither convex nor differentiable in $(v,\dot{v})$. In the search for local minima of the integral I over E, the classical Euler-Lagrange and transversality conditions will not be applicable. However, the necessary conditions of Clarke [1] will be available. We state:

Proposition 2.2. Assume v is a local minimum of the dual action integral I over E. Then there is some vector $\xi \in \mathbb{R}^{2n}$ such that the translate u defined by $u(t) = v(t) + \xi$ is a solution of the Hamiltonian system (H) on the time interval $[0,T]$ satisfying the boundary condition $u(0) = u(T)$.

Proof. Let v be a local minimum for I over E. Using the terminology of [1], this implies that v is a weak local minimum of the variational problem:

$$\left\{ \begin{aligned} & \operatorname{Inf} \int_0^T L(t,v(t),\dot{v}(t))dt \\ & v(0) = v(T) \end{aligned} \right. \tag{16}$$

The integrand L is given in formula (15); it is locally Lipschitz in $(v,\dot{v})$, so that the necessary conditions of [1] hold. They tell us that there exists an absolutely continuous function $\lambda : [0,T] \to \mathbb{R}^{2n}$ such that:

$$\dot{\lambda}(t) = \frac{1}{2}\sigma\dot{v}(t) \quad \text{a.e.} \tag{17}$$

$$\lambda(t) = -\frac{1}{2}\sigma v(t) + \sigma\partial G(t,-\sigma\dot{v}(t) + \sigma f(t)) \quad \text{a.e.} \tag{18}$$

$$\lambda(0) = \lambda(T) \tag{19}$$

In the particular case when G is C^1, differentiating (18) with respect to time and comparing with (17) yields the usual Euler-Lagrange equations, while (19) is the usual transversality condition.

We then define a function $u : [0,T] \to \mathbb{R}^n$ by:

$$u(t) = \frac{1}{2}v(t) - \sigma\lambda(t) \tag{20}$$

Equation (17) tells us that $\dot{u} = \dot{v}$, so that $u = v + \xi$ for some constant $\xi \in \mathbb{R}^{2n}$. Equation (18) we rewrite as follows:

$$u(t) \in \partial G(t,-\sigma\dot{v}(t) + \sigma f(t)) \quad \text{a.e.} \tag{21}$$

We invert this equation by Fenchel's reciprocity formula (10) $\Leftrightarrow$ (11):

$$-\sigma\dot{v}(t) + \sigma f(t) \in \partial H(t,u(t)) \quad \text{a.e.} \tag{22}$$

Since $\dot{u} = \dot{v}$, this gives equation ($\mathcal{H}$) for u:

$$\dot{u}(t) \in \sigma\partial H(t,u(t)) + f(t) \quad \text{a.e.} \tag{23}$$

Finally, conditions (19) and (20) yield $u(0) = u(T)$.

§III - PERIODIC SOLUTIONS.

We recall the differential system we are dealing with:

$$(\mathcal{H}) \qquad \dot{u}(t) \in \sigma\partial H(t,u(t)) + f(t) \quad \text{a.e.}$$

under the assumptions that $H(t,u)$ is measurable in t, convex continuous in u, and

$$(1) \qquad H(t,u) \geq H(t,0) = 0 \quad \text{all } (t,u)$$

$$(2) \qquad H(t,u) \leq \frac{k^\beta}{\beta} |u|^\beta \quad \text{with } k > 0 \text{ and } \beta > 2$$

$$(3) \qquad \lim_{r\to\infty} \operatorname{Min}_{|u|=r} H(t,u) = +\infty$$

From now on, the forcing term f and the Hamiltonian $H(\cdot,u)$ (for each fixed $u \in \mathbb{R}^{2n}$) are assumed to be T-periodic functions of t, for some given $T > 0$. We will show that, if the forcing term f is not too large, and has mean zero, there is a T-periodic solution to system $(\mathcal{H})$.

Recall that α is the conjugate exponent of β, i.e. $\alpha^{-1} + \beta^{-1} = 1$. We then introduce some constants, the actual values of which can be computed from β:

$$(4) \qquad b(\beta) = (2\pi)^{-2/\beta}$$

$$(5) \qquad c(\beta) = (2-\alpha)(2\alpha-2)^{\frac{\alpha-1}{2-\alpha}}(\alpha b(\beta))^{\frac{-1}{2-\alpha}}$$

$$(6) \qquad d(\beta) = \left(\frac{2}{\beta}\right)^{\frac{1}{2-\alpha}} b(\beta)^{\frac{-1}{2-\alpha}}$$

<u>Proposition 3.1</u>. Assume $\|f\|_\alpha \leq c(\beta)k^{\frac{-\beta}{\beta-2}} T^{-\frac{2}{\beta}\frac{(\beta-1)}{(\beta-2)}}$, and $\int_0^T f(t)dt = 0$. Consider the ball B in E defined by:

$$(7) \qquad v \in B \Leftrightarrow \|\dot{v} - f\|_\alpha \leq d(\beta)k^{\frac{-\beta}{\beta-2}} T^{-\frac{2}{\beta}\frac{(\beta-1)}{(\beta-2)}}.$$

The dual action integral I then attains its minimum relative to B. Any point $v \in B$ where this minimum is attained satisfies the sharper estimate:

$$(8) \qquad \|\dot{v} - f\|_\alpha \leq \frac{d(\beta)}{c(\beta)} \|f\|_\alpha = \frac{2}{\beta-2} \|f\|_\alpha .$$

Before starting the proof, we bring the origin to f in the space E, so as to get more convenient expressions for the dual action integral I and the ball B. We set $\dot{w} = \dot{v} - f$, and hence $w = v - F$, where $F(t)$ is the primitive of $f(t)$ with mean zero:

$$\dot{F}(t) = f(t) \quad \text{and} \quad \int_0^T F(t)dt = 0 \ . \tag{9}$$

The ball B now is defined by $\|\dot{w}\| \leq d(\beta)k^{\frac{-\beta}{\beta-2}} T^{-\frac{2}{\beta}\frac{(\beta-1)}{(\beta-2)}}$. The functional I becomes, in the new coordinates for E:

$$\begin{aligned} I(w) &= \int_0^T \left\{ \frac{1}{2}(\sigma\dot{w}(t) + \sigma f(t), w(t) + F(t)) + G(t,-\sigma\dot{w}(t)) \right\} dt \\ &= \left\{ \frac{1}{2}\langle \sigma f, F \rangle + \langle \sigma\dot{w}, F \rangle + \frac{1}{2}\langle \sigma\dot{w}, w \rangle + \int_0^T G(t,-\sigma\dot{w}(t)) \right\} dt \ . \end{aligned}$$

The brackets denote the duality pairing between $L^{\alpha}(0,T;\mathbb{R}^{2n})$ and $L^{\beta}(0,T;\mathbb{R}^{2n})$. The first term, a constant, can be disregarded for minimization purposes. The functional to investigate thus becomes:

$$\bar{I}(w) = \langle \sigma\dot{w}, F \rangle + \frac{1}{2}\langle \sigma\dot{w}, w \rangle + \int_0^T G(t,-\sigma\dot{w}(t))dt \tag{10}$$

We now proceed in two steps:

<u>Step 1</u>. $\bar{I}(w) > 0$ for all $w \in \partial B$.

We estimate I from below. Formula (10) yields:

$$\bar{I}(w) \geq -\|\sigma\dot{w}\|_{\alpha}\|F\|_{\beta} - \frac{1}{2}\|\sigma\dot{w}\|_{\alpha}\|w\|_{\beta} + \int_0^T G(t,-\sigma\dot{w}(t))dt \ . \tag{11}$$

Using lemma 2.1, we go further:

$$\bar{I}(w) \geq -\|\sigma\dot{w}\|_{\alpha}\|F\|_{\beta} - \frac{1}{2}\|\sigma\dot{w}\|_{\alpha}\|w\|_{\beta} + \frac{1}{\alpha k^{\alpha}}\|\sigma\dot{w}\|_{\alpha}^{\alpha} \ . \tag{12}$$

Because of formula (2.5), we see that σ is an isometry, so that $\|\sigma\dot{w}\|_{\alpha} = \|\dot{w}\|_{\alpha}$. The evaluation of $\|w\|_{\beta}$ in terms of $\|\dot{w}\|_{\alpha}$ and of $\|F\|_{\beta}$ in terms of $\|f\|_{\alpha}$ presents us with a special problem.

The map $\dot{w} \to w$, with $\int_0^T w(t)dt = 0$, from L^α to L^β, is a linear continuous operator. When $\alpha = 1$ and $\beta = \infty$, it is readily seen to have norm 1. When $\alpha = 2$ and $\beta = 2$, it is known to have norm $\frac{T}{2\pi}$ (see [3] or [4]). Using the convexity theorem of Marcel Riesz (see [5] for instance), we conclude that in the case where $1 \leq \alpha \leq 2$, its norm is at most $\exp|-2\beta^{-1}\mathrm{Log}\left(\frac{2\pi}{T}\right)|$, which is $b(\beta)T^{2/\beta}$.

Inequality (12) now becomes:

$$\bar{I}(w) \geq -b(\beta)T^{2/\beta}\|f\|_\alpha\|\dot{w}\|_\alpha - \frac{1}{2}\,b(\beta)T^{2/\beta}\|\dot{w}\|_\alpha^2 + \frac{1}{\alpha k^\alpha}\|\dot{w}\|_\alpha^\alpha . \tag{13}$$

Now consider the function $\varphi(s)$ of the real variable s:

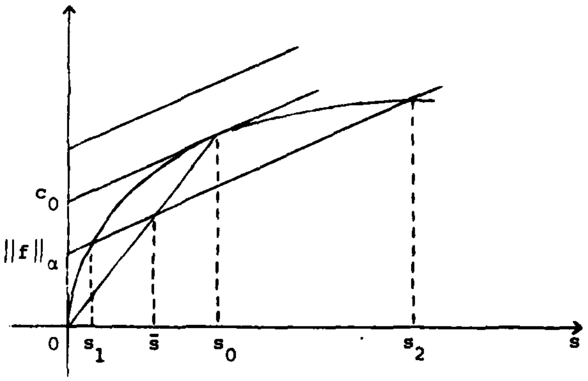
$$\varphi(s) = -b(\beta)T^{2/\beta}\|f\|_\alpha s - \frac{1}{2}\,b(\beta)T^{2/\beta}s^2 + \frac{1}{\alpha k^\alpha}\,s^\alpha \tag{14}$$

Clearly $\varphi(0) = 0$, and $\varphi'(0) < 0$. We want to solve the equation $\varphi(s) = 0$ with $s > 0$. After simplifying by s, this becomes:

$$\|f\|_\alpha + \frac{1}{2}\,s = \frac{1}{\alpha k^\alpha}\,\frac{T^{-2/\beta}}{b(\beta)}\,s^{\alpha-1} . \tag{15}$$

In other words, we seek the intersection of the curve $s \to \frac{1}{\alpha k^\alpha}\frac{T^{-2/\beta}}{b(\beta)}\,s^{\alpha-1}$ with the straight line $s \to \|f\|_\alpha + \frac{1}{2}\,s$. This is easily done. We first seek out the point s_0 on the curve where the slope of the tangent is $\frac{1}{2}$; there will be two points of intersection s_1 and s_2 if the given line lies under the tangent, none if it lies above:

(16)



The computation gives:

(17) $$\frac{T^{-2/\beta}}{b(\beta)} \frac{\alpha-1}{\alpha k^{\alpha}} s_0^{\alpha-2} = \frac{1}{2}, \quad \text{hence} \quad s_0 = \left|\frac{\alpha k^{\alpha} b(\beta)}{2(\alpha-1)}\right|^{\frac{1}{\alpha-2}} T^{\frac{2}{\beta(\alpha-2)}}$$

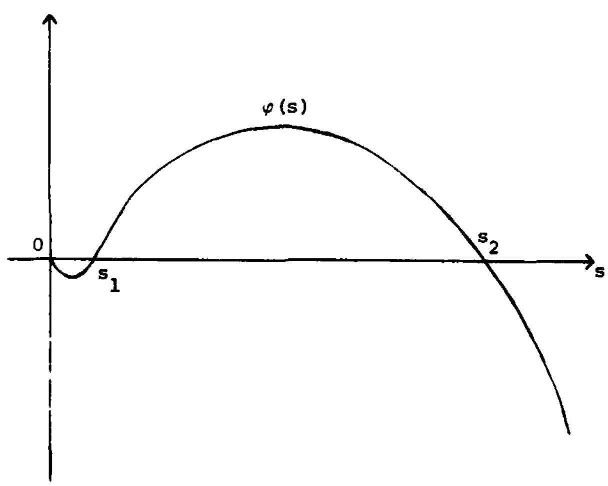
(18) $$c_0 = \frac{1}{\alpha k^{\alpha}} \frac{T^{-2/\beta}}{b(\beta)} s_0^{\alpha-1} - \frac{\alpha-1}{\alpha k^{\alpha}} \frac{T^{-2/\beta}}{b(\beta)} s_0^{\alpha-1} = \frac{2-\alpha}{\alpha k^{\alpha}} \frac{T^{-2/\beta}}{b(\beta)} s_0^{\alpha-1}$$

The equation $\varphi(s) = 0$ will have two different roots $s_2 > s_1 > 0$ provided that $\|f\|_{\alpha} \leq c_0$, that is:

(19) $$\|f\|_{\alpha} \leq (2-\alpha)(\alpha b(\beta))^{\frac{1}{\alpha-2}} (2\alpha - 2)^{\frac{\alpha-1}{\alpha-2}} k^{\frac{\alpha}{\alpha-2}} T^{\frac{2}{\beta(\alpha-2)}}$$

The graph of φ then looks like:

(20)



The result now follows immediately from the estimate $\bar{I}(w) \geq \varphi(\|w\|_{\alpha})$, and the fact that s_0 lies between s_1 and s_2.

Step 2. $\bar{I}$ attains its minimum on B.

Let w_n be a minimizing sequence in B:

(21) $$\bar{I}(w_n) \to \operatorname{Inf}\{\bar{I}(w) \mid w \in B\} .$$

The sequence $\dot{w}_n$ belongs to the ball with radius $d'(\beta)k^{\frac{-\beta}{\beta-2}}T^{-\frac{2}{\beta}\frac{(\beta-1)}{(\beta-2)}}$ and center 0 in L^α. Since this space is reflexive, there is a subsequence $w_{n'}$ which converges weakly to some w belonging to the same ball:

(22) $$\dot{w}_{n'} \rightharpoonup \dot{w} \text{ weakly in } L^\alpha$$

(23) $$w_{n'} \to w \text{ strongly in } L^\beta$$

(24) $$w \in B .$$

The strong convergence of the $w_{n'}$ follows from the compactness of the map $\dot{w} \to w$ from L^α to L^β. The weak convergence (22) yields immediately:

(25) $$\langle \sigma\dot{w}_{n'}, F\rangle \to \langle \sigma\dot{w}, F\rangle$$

Moreover, we have:

(26) $$\langle \sigma\dot{w}_{n'}, w_{n'}\rangle - \langle \sigma\dot{w}, w\rangle = \langle \sigma\dot{w}_{n'} - \sigma\dot{w}, w\rangle + \langle \sigma\dot{w}_{n'}, w_{n'} - w\rangle .$$

The first term goes to zero because the $\dot{w}_{n'}$ converge weakly, and the second one goes to zero because the $\dot{w}_{n'}$ are uniformly bounded and the $w_{n'}$ converge strongly:

(27) $$\frac{1}{2}\langle \sigma\dot{w}_{n'}, w_{n'}\rangle \to \frac{1}{2}\langle \sigma\dot{w}, w\rangle$$

Finally, by known properties of non-negative convex integrands (see [7] or [11]), we have:

(28) $$\liminf \int_0^T G(t,-\sigma\dot{w}_{n'}(t))dt \geq \int_0^T G(t,-\sigma\dot{w}(t))dt .$$

Adding up (25), (27), (28), and comparing them with formulas (10) and (21), we get:

(29) $$\bar{I}(w) \leq \operatorname{Inf}\{\bar{I}(w') \mid w' \in B\} .$$

Since $w \in B$, equality must hold in (29), proving that w is a minimizer in B.

<u>Conclusion</u>. The minimizers satisfy estimate (8).

From formula (10), we see that $\bar{I}(0) = 0$. Since $0 \in B$, we see that $I(w) \leq 0$ for any minimizer w of $\bar{I}$ on B. This implies that $\|w\|_\alpha$ must lie between 0 and the first positive root s_1 of φ. Figure (16) gives us by inspection the desired estimate:

(30) $$s_1 < \bar{s} = s_0 \frac{\|f\|_\alpha}{c_0} .$$

Corollary 3.2. Assume f and $H(\cdot,u)$ are T-periodic, with:

$$\int_0^T f(t)dt = 0 \quad \text{and} \quad \|f\|_\alpha \le c(\beta)k^{\frac{-\beta}{\beta-2}} T^{-\frac{2}{\beta}\frac{(\beta-1)}{(\beta-2)}} \tag{31}$$

Then the Hamiltonian system (H) has at least one T-periodic solution u such that:

$$\|\dot{u} - f\|_\alpha \le \frac{2}{\beta-2}\|f\|_\alpha . \tag{32}$$

Proof. In proposition 3.1, we have found some $v \in E$ which minimizes I on B, and which is interior to B because of estimate (8). Clearly v is a local minimum for $\bar{I}$ on E, so that we can apply proposition 2.2. The result follows immediately; estimate (32) follows from (8) and the relation $\dot{u} = \dot{v}$. /

We will refer to the T-periodic solutions found in this way as solutions of type (E). When there is no forcing term, $f = 0$, this type (E) solution is simply $u = 0$, rest at equilibrium. When the forcing term f is small, estimate (31) tells us that the solution u is almost constant. With a few more assumptions on H, and the equation $u \in \partial G(-\sigma\dot{u} + \sigma f)$, it can in fact be proved that u is small. For instance:

Corollary 3.3. Assume moreover that there is some constant $c > 0$ such that $|v| \ge c|u|^{\beta-1}$ for all $v \in \partial H(t,u)$. Then, in addition to (32) the T-periodic solutions of type (E) satisfy the following estimate:

$$\|u\|_\beta \le \left|\frac{1}{c}\,\frac{2}{\beta-2}\,\|f\|_\alpha\right|^{\frac{1}{\beta-1}} . \tag{33}$$

Proof. We have $-\sigma(\dot{u}(t) - f(t)) \in \partial H(t,u(t))$, so that:

$$|u(t)|^\beta \le c^{\frac{-\beta}{\beta-1}} |\dot{u}(t) - f(t)|^{\frac{\beta}{\beta-1}} \tag{34}$$

Integrating over $[0,T]$ yields the desired result. /

We conclude this argument with two remarks. First, note that the estimate (8) is very rough, and more elaborate calculations will yield better ones. For instance, it is clear from figure (16) that $s_1 = O(\|f\|_\alpha^{1/(\alpha-1)})$ (using Landau's symbol O), so that, when $\|f\|_\alpha \to 0$, we have the estimate $\|\dot{u} - f\|_\alpha = O(\|f\|_\alpha^{\beta-1})$, which is certainly better than (32).

Note also that the preceding argument will carry over, with suitable modifications, to the case $\beta = 2$. However, we then fall within the scope of the paper [4], to which we refer for results.

IV - OTHER HAMILTONIANS.

We now wish to extend the preceding results to other Hamiltonians, which do not satisfy the inequality $H(t,u) \leq k^{\beta}|u|^{\beta}\beta^{-1}$ over all of $\mathbb{R}^{2n}$.

We begin with Hamiltonians which satisfy this condition in a neighbourhood of the origin only. For the sake of simplicity, we shall assume that the Hamiltonian H does not depend on t. Throughout, we assume that $H'(0,0) = 0$.

<u>Proposition 4.1</u>. Assume that there is a neighbourhood $\mathcal{U}$ of the origin in $\mathbb{R}^{2n}$ such that H is C^2 on $\mathcal{U}$, the second derivative $H''(u)$ being positive definite for $u \neq 0$, and satisfying, for some constants $b > a > 0$ and $\beta > 2$:

(1) $$a|u|^{\beta-2}|v|^2 \leq (H''(u)v,v) \leq b|u|^{\beta-2}|v|^2, \quad \text{all } u \in \mathcal{U},\ v \in \mathbb{R}^{2n} .$$

Then, for any $T > 0$, there is some $\varepsilon > 0$ such that, whenever $\|f\|_{\alpha} \leq \varepsilon$, with $\int_0^T f dt = 0$, the Hamiltonian system $(\mathcal{H})$ has some periodic solution lying inside $\mathcal{U}$.

<u>Proof</u>. We can always assume that $\mathcal{U}$ is a ball with radius $\eta > 0$. It follows from the assumptions that H is convex on $\mathcal{U}$. Now consider a ray $t \to tu$ from the origin, with $|u| = 1$. As long as $0 < t < \eta$, we have $tu \in \mathcal{U}$, and, setting $\gamma = \beta - 2$:

$$(H'(tu),v) = \int_0^t (H''(su)u,v)ds$$

$$\geq \int_0^t as^{\gamma}|v|ds = \frac{a}{\gamma+1} t^{\gamma+1}|v|$$

$$\leq \int_0^t bs^{\gamma}|v|ds = \frac{b}{\gamma+1} t^{\gamma+1}|v| .$$

Hence, for all $u \in \mathcal{U}$:

(2) $$a(\gamma + 1)^{-1}|u|^{\gamma+1} \leq \|H'(u)\| \leq b(\gamma + 1)^{-1}|u|^{\gamma+1}$$

Integrating once more yields, for all $u \in \mathcal{U}$:

(3) $$a \frac{|u|^{\gamma+2}}{(\gamma+1)(\gamma+2)} \leq H(u) \leq b \frac{|u|^{\gamma+2}}{(\gamma+1)(\gamma+2)}$$

It is now simply a matter of finding a convex function $\bar{H} : \mathbb{R}^n \to \mathbb{R}$, satisfying (2) and (3) over all of $\mathbb{R}^n$, and coinciding with H on $\mathcal{U}$. This being done, we apply

to $\overline{H}$ corollaries 3.2 and 3.3, with $\beta = \gamma + 2$. It follows from estimates (3.32) and (3.33) that if $\|f\|_\alpha$ is small enough, the periodic solution u we have found for

$$\dot{u}(t) = \sigma\overline{H}'(u(t)) + f(t) \tag{4}$$

will lie entirely inside U, so that $\overline{H}'(u(t)) = H'(u(t))$ for all t. It follows that it is actually a solution of:

$$\dot{u}(t) = \sigma H'(u(t)) + f(t) \ . \ / \tag{5}$$

We now turn to another class of Hamiltonians, of particular importance for applications. These are the Hamiltonians which split as:

$$H(t,x,p) = \frac{1}{2} p^2 + V(t,x) \ . \tag{6}$$

Such Hamiltonians are common in classical mechanics. The first term is kinetic energy, the second one potential energy. Because the first term is quadratic, they cannot satisfy growth conditions such as (2.2)-(3.2), even locally.

However, results similar to proposition 3.1 and its corollaries still hold, with slight modifications. The growth assumptions now will be made directly on the potential $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. We shall assume it to be:

$$\text{measurable in } t, \text{ convex continuous in } x \tag{7}$$

$$V(t,x) \geq V(t,0) = 0, \quad \text{all } (t,x) \tag{8}$$

$$V(t,x) \leq \frac{k^\beta}{\beta} |x|^\beta, \quad \text{all } (t,x), \text{ some } k > 0 \text{ and } \beta > 2 \tag{9}$$

$$\forall t, \quad \underset{|x|=r}{\text{Min}} \ V(t,x) \to +\infty \ \text{ when } \ r \to \infty \tag{10}$$

We introduce new constants:

$$c'(\beta) = (2 - \alpha)(\alpha\pi^{-2/\beta}b(\beta)2^{-1})^{\frac{1}{\alpha-2}}(2\alpha - 2)^{-\frac{\alpha-1}{\alpha-2}} \tag{11}$$

$$d'(\beta) = \left|\frac{4}{\beta}\frac{\pi^{2/\beta}}{b(\beta)}\right|^{\frac{1}{2-\alpha}} \tag{12}$$

<u>Proposition 4.2</u>. Assume V satisfies conditions (7) to (10), and:

$$\int_0^T f(t)dt = 0 \quad \text{and} \quad \|f\|_\alpha \leq c'(\beta)k^{\frac{-\beta}{\beta-2}}T^{-\frac{(\beta+2)(\beta-1)}{\beta(\beta-2)}} \ . \tag{13}$$

Then the Hamiltonian system:

(14)
$$(\dot{x},\dot{p}) \in (p,-\partial V(t,x)) + (0,f(t))$$

has at least one solution (x,p) such that:

(15)
$$x(0) = x(T) \quad \text{and} \quad p(0) = p(T)$$

(16)
$$\|\dot{p} - f\|_\alpha \leq \frac{d'(\beta)}{c'(\beta)} \|f\|_\alpha = \frac{2}{\beta - 2} \|f\|_\alpha$$

<u>Proof</u>. Denote by $v = (y,q)$ the dual variable of $u = (x,p)$. We can easily compute $G(y,q)$ for this case:

(17)
$$\begin{aligned} G(y,q) &= \sup_{x,p} \{xy + pq - \frac{1}{2} p^2 - V(t,x)\} \\ &= \sup_{x} \{xy - V(t,x)\} + \sup_{p} \{pq - \frac{1}{2} p^2\} \\ &= V^*(t,y) + \frac{1}{2} q^2 . \end{aligned}$$

Here $V^*(t,\cdot)$ is the Legendre transform of the function $V(t,\cdot)$. It is convex, continuous, minimum at the origin, and satisfies the estimate:

(18)
$$V^*(t,u) \geq \frac{1}{\alpha k^\alpha} |y|^\alpha$$

The dual action integral now is:

(19)
$$I(y,q) = \int_0^T \{-\dot{y}q + \frac{1}{2} \dot{y}^2 + V^*(t,-\sigma\dot{q} + \sigma f)\}dt$$

on the space E defined by:

(20)
$$\dot{y} \in L^2(0,T;\mathbb{R}^n), \int_0^T \dot{y}dt = 0 = \int_0^T ydt$$

(21)
$$\dot{q} \in L^\alpha(0,T;\mathbb{R}^n), \int_0^T \dot{q}dt = 0 = \int_0^T qdt .$$

We wish to prove that the functional I has a local minimum on E. For this, we have to estimate it from below. We first write it slightly differently:

(22)
$$\begin{aligned} I(y,q) &= \frac{1}{2} \int_0^T (\dot{y} - q)^2 dt + \int_0^T \{-\frac{1}{2}q^2 + V^*(t,-\dot{q} + f)\}dt \\ &= \frac{1}{2} \|\dot{y} - q\|_2^2 + J(q) . \end{aligned}$$

Here

$$J(q) = \int_0^T \{-\frac{1}{2}q^2 + V^*(t,-\dot{q} + f)\}dt \; . \tag{23}$$

Using inequality (18) we get an estimate for J:

$$J(q) \geq -\frac{1}{2}\|q\|_2^2 + \frac{1}{\alpha k^\alpha}\|\dot{q} - f\|_\alpha^\alpha . \tag{24}$$

Call F the primitive of f which has mean value zero, and set $q' = q - F$. We get, denoting by brackets the duality pairing between $L^\alpha(0,T;\mathbb{R}^n)$ and $L^\beta(0,t;\mathbb{R}^n)$:

$$J(q) \geq -\frac{1}{2}\langle F,F\rangle + \langle F,q'\rangle - \frac{1}{2}\langle q',q'\rangle + \frac{1}{\alpha k^\alpha}\|\dot{q}'\|_\alpha^\alpha . \tag{25}$$

To go further, we need to estimate $\|q'\|_\alpha$ and $\|F\|_\alpha$ in terms of $\|\dot{q}'\|_\alpha$ and $\|f\|_\alpha$. Taking into account the normalization conditions, $\int_0^T q'dt = 0 = \int_0^T \dot{q}'dt$, we get $\|q'\|_2 \leq \frac{T}{2\pi}\|\dot{q}'\|_2$ and $\|q'\|_\infty \leq \frac{T}{2}\|\dot{q}'\|_\infty$. Using the convexity theorem of M. Riesz again, we get $\|q'\|_\beta \leq \frac{T}{2}\pi^{-2/\beta}$ for $2 \leq \beta \leq \infty$. Transposing, we get, for $1 < \alpha < 2$:

$$\|q'\|_\alpha \leq \frac{T}{2}\pi^{-2/\beta}\|\dot{q}'\|_\alpha \tag{26}$$

Similarly

$$\|F\|_\alpha \leq \frac{T}{2}\pi^{-2/\beta}\|f\|_\alpha \tag{27}$$

Writing this back into inequality (25) yields:

$$J(q) \geq \psi(\|\dot{q}'\|_\alpha) - \frac{1}{2}\langle F,F\rangle \tag{28}$$

where the function ψ of the real variable s is given by:

$$\psi(s) = -\frac{T}{2}\pi^{-2/\beta}\|f\|_\alpha T^{2/\beta}b(\beta)s - \frac{1}{2}\frac{T}{2}\pi^{-2/\beta}T^{2/\beta}b(\beta)s^2 + \frac{1}{\alpha k^\alpha}s^\alpha \tag{29}$$

$$= -\pi^{-2/\beta}\frac{b(\beta)}{2}T^{\frac{\beta+2}{\beta}}\|f\|_\alpha s - \frac{1}{2}\pi^{-2/\beta}\frac{b(\beta)}{2}T^{\frac{\beta+2}{\beta}}s^2 + \frac{1}{\alpha k^\alpha}s^\alpha$$

This is the same function as $\varphi(s)$ of formula (3.14), with the coefficient $b(\beta)$ changed to $\pi^{-2/\beta}b(\beta)/2$ and the exponent $2/\beta$ changed to $(\beta + 2)\beta^{-1}$. It follows that, provided:

$$\|f\|_\alpha \leq (2 - \alpha)(\alpha\pi^{-2/\beta}b(\beta)2^{-1})^{\frac{1}{\alpha-2}}(2\alpha - 2)^{-\frac{\alpha-1}{\alpha-2}}k^{\frac{\alpha}{\alpha-2}}T^{\frac{\beta+2}{\beta(\alpha-2)}} \tag{30}$$

we will have:

$$\varphi(s_0') > 0, \quad \text{with} \quad s_0' = \left(\frac{\alpha k^{\alpha,-2/\beta} b(\alpha)}{4(\alpha - 1)}\right)^{\frac{1}{\alpha-2}} T^{\frac{\beta+2}{\beta(\alpha-2)}} \tag{31}$$

From then on, we proceed as in the proof of proposition 3.1. Going back to equation (22), we have:

$$I(y,q) \geq \frac{1}{2}\|\dot{y} - q\|_2^2 + \varphi(\|\dot{q} - f\|_\alpha) \ . \tag{32}$$

Provided condition (30) is satisfied, the functional I will attain its minimum relative to the cylinder C in E defined by:

$$C = \{(y,q) \in E \mid \|\dot{q} - f\|_\alpha \leq s_0'\} \tag{33}$$

We show as in proposition 3.1 that this minimum is attained at an interior point, indeed that condition (16) holds. The proof of proposition 2.2 then carries over to this case, showing that some translate (x,p) of (y,q) is a T-periodic solution of the Hamiltonian system (14). /

Of course, in the case where the potential $V(t,x)$ is differentiable with respect to x, a more compact way to write system (14) is Newton's equation:

$$\ddot{x} + V'(t,x) = f(t) \ . \tag{34}$$

Let us give a simple example to illustrate proposition 4.2. The n-dimensional system of differential equations:

$$\ddot{x}_i + ax_i \sum_{j=1}^{n} x_j^2 = f_i(t) \qquad 1 \leq i \leq n, \quad a > 0 \tag{35}$$

will have a T-periodic solution, provided the forcing terms f_i all are T-periodic, have mean zero, and satisfy:

$$\|f\|_{4/3} = \left|\int_0^T \left(\sum_{i=1}^{n} f_i(t)^2\right)^{2/3} dt\right|^{3/4} \leq c'(4)a^{-1/2}T^{-9/4} \tag{36}$$

This solution will satisfy the estimate:

$$\|\ddot{x} - f\|_{4/3} \leq \|f\|_{4/3} \ . \tag{37}$$

Another estimate follows immediately by substituting equation (35):

$$\|x\|_4 = \left|\int_0^T \left(\sum_{i=1}^n x_i(t)^2\right)^2 dt\right|^{1/4} \leq \left(\frac{1}{a}\|f\|_{4/3}\right)^{1/3} \tag{38}$$

All this follows from proposition 4.2, with the exponent $\beta = 4$ and the potential $V(x) = a4^{-1}\left(\sum_{i=1}^n x_i^2\right)^2$. Here $c'(4) = d'(4) = (\pi\sqrt{2})^{3/2} \simeq 6.06$.

Finally, we can adapt the proof of proposition 4.1 to the particular case of classical Hamiltonians, to get the following result:

Proposition 4.3. Assume $V'(0) = 0$ and there is a neighbourhood U of the origin in $\mathbb{R}^n$ such that v is C^2 on U, the second derivative $V''(x)$ being positive definite for $x \neq 0$, and satisfying, for some constants $b' > a' > 0$ and $\beta > 2$:

$$a'|x|^{\beta-2}|y|^2 \leq (V''(x)y,y) \leq b'|x|^{\beta-2}|y|^2, \quad \text{all } x \in U,\ y \in \mathbb{R}^n \tag{39}$$

Then for any $T > 0$, there is some $\varepsilon > 0$ such that, whenever $\|f\|_\alpha \leq \varepsilon$, with $\int_0^T f dt = 0$, the Hamiltonian system:

$$\ddot{x} \in \partial V(x) + f(t) \quad \text{a.e.} \tag{40}$$

has some periodic solution lying inside U.

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We study periodic solutions of the nonlinear Hamiltonian system with n degrees of freedom

$$(H)\qquad \begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i}(x,p) + g_i(t) \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i}(x,p) + h_i(t) \end{aligned}$$

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ABSTRACT (continued)

the Hamiltonian H being convex and super quadratic in both variables, and the forcing terms being T-periodic with mean value zero. We prove that, if these forcing terms lie within bounds which we explicitly compute, system (H) has some T-periodic solution, which we also locate explicitly.